Math 206B Lecture 19 Notes

Daniel Raban

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1 Bases, Involution, and Scalar Product of Symmetric Functions

1.1 Five bases of symmetric functions

Let $\Lambda = \varprojlim_n \lambda_n$ be the ring of symmetric functions. Then $\Lambda \subseteq \mathbb{C}[x_1, x_2, \dots]$; that is $f \in \Lambda = \sum c_\alpha x^\alpha$, where $\alpha = (\alpha_1, \alpha_2, \dots)$, $\sum_\alpha \alpha_i < \infty$, and $\alpha_i \in \mathbb{N}$.

Example 1.1. $e_2 = x_1x_2 + x_1x_3 + x_2x_3 + x_1x_4 + x_2x_4 + x_3x_4 + x_1x_5 + \cdots$.

Definition 1.1. The **monomial symmetric functions** are

$$m_\lambda = \left[\sum_{\substack{\sigma \in S_\ell \\ i_1 < \cdots < i_\ell}} x_{i_1}^{\lambda_{\sigma(1)}} x_{i_2}^{\lambda_{\sigma(2)}} \cdots x_{i_\ell}^{\lambda_{\sigma(\ell)}} \right] / \prod_{i=1}^{\ell} m_i(\lambda)!,$$

where $\lambda = (\lambda_1, \dots, \lambda_\ell)$.

Proposition 1.1. *The monomial symmetric functions form a basis for Λ.*

Definition 1.2. The **elementary symmetric functions** are

$$e_k = m_{(1^k)} = \sum_{i_1 < \cdots < i_k} x_{i_1} \cdots x_{i_k},$$

$$e_\lambda = e_{\lambda_1} \cdots e_{\lambda_\ell},$$

where $\lambda = (\lambda_1, \dots, \lambda_\ell)$.

Proposition 1.2. *The elementary symmetric functions form a basis for Λ.*

Proof. $f \in \Lambda$ is $c_\lambda x^\lambda + \cdots$, when written in lexicographic order. $\square$

Theorem 1.1. *The elementary symmetric functions are free generators of Λ as a ring; i.e. they do not satisfy any algebraic equations.*

Definition 1.3. The **power symmetric functions** are

$$p_k = m_{(k)} = x_1^k + x_2^k + \cdots ,$$

$$p_\lambda = p_{\lambda_1} \cdots p_{\lambda_\ell},$$

where $\lambda = (\lambda_1, \ldots, \lambda_\ell)$.

Proposition 1.3. *The power symmetric functions form a basis for* Λ*.*

Definition 1.4. The **complete symmetric functions** are

$$h_k = \sum_{|\lambda|=k} m_\lambda = \sum_{i_1 \leq i_2 \leq \cdots \leq i_k} x_{i_1} x_{i_2} \cdots x_{i_k},$$

$$h_\lambda = h_{\lambda_1} \cdots h_{\lambda_\ell},$$

where $\lambda = (\lambda_1, \ldots, \lambda_\ell)$.

Proposition 1.4. *The complete symmetric functions form a basis for* Λ*.*

Definition 1.5. The **Schur functions** are

$$s_\lambda = \frac{a_{\lambda+\rho}}{a_\rho} = \sum_{A \in \mathrm{SSYT}(\lambda)} x^A,$$

$$x^A = x_1^{m_1(A)} x_2^{m_2(A)} \cdots .$$

Theorem 1.2. *The Schur functions form a basis for* Λ*.*

1.2 Involution and scalar product on symmetric functions

Here is a dictionary relating symmetric functions and representation theory of S_n

Symmetric functions	Representations of S_n
s_λ	S^λ
h_λ	$M^\lambda = \mathrm{ind}_{S_{\lambda_1} \times \cdots \times a_{\lambda_\ell}}^{S_n} 1$
e_λ	$M^\lambda \otimes \mathrm{sgn}$

This correspondence tells that we should have an involution $\omega : \Lambda \to \Lambda$ sending $e_\lambda \mapsto h_\lambda$ corresponding to $\otimes\,\mathrm{sgn}$.

Theorem 1.3. *The involution* $\omega : \Lambda \to \Lambda$ *sends* $s_\lambda \mapsto s_{\lambda'}$*.*

Theorem 1.4. *The involution* $\omega : \Lambda \to \Lambda$ *sends* $p_\lambda \mapsto \varepsilon_\lambda p_\lambda$*.*

There is a scalar product on Λ that relates to the scalar product on characters of representations of S_n.

Definition 1.6. Define a **scalar product** on Λ by its value on the basis of Schur functions:

$$\langle s_\lambda, s_\mu \rangle = \delta_{\lambda,\mu}.$$

If $f = \sum c_\lambda s_\lambda$ and $g = \sum r_\lambda s_\lambda$, then

$$\langle f, g \rangle = \sum_\lambda c_\lambda r_\lambda.$$

Proposition 1.5. *$\langle m_\lambda, h_\mu \rangle = \delta_{\lambda,\mu}$ for all λ, μ.*

Proof. Write $M^\mu = \bigoplus K_{\lambda,\mu} S^\lambda$. Then $h_\mu = \sum K_{\lambda,\mu} s_{\lambda,\mu}$. We also have $s_\lambda = \sum K_{\lambda,\mu} m_\mu$. □

Proposition 1.6. *$\langle p_\lambda, p_\mu \rangle = z_\lambda \delta_{\lambda,\mu}$, where*

$$z_\lambda = \frac{n!}{1^{m_1} m_1! 2^{m_2} m_2! \cdots}$$

is the size of the conjugacy class corresponding to λ in S_n.

Theorem 1.5. $s_\lambda = \sum_\mu \chi_\lambda[\mu] p_\mu$.[1]

So we can talk about characters of the symmetric group by only talking about symmetric functions.[2]

[1] Maybe there is a factor of z_λ in here. Professor Pak doesn't remember.
[2] Newton studied p_μ. That's how old the idea of symmetric functions is.